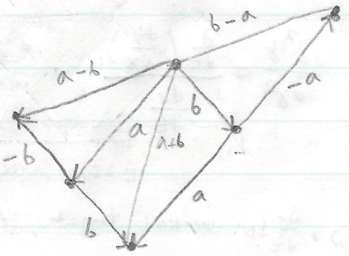
7 questions, mostly chp 14
2 theoretical q's.

Mat232 Formula Sheet



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $\langle b_1-a_1, b_2-a_2 \rangle$ for $A=(a_1,a_2)$ $B=(b_1,b_2)$

$\vec{A}+\vec{B} = \langle a_1+b_1, a_2+b_2 \rangle$

$|A| = \sqrt{a_1^2+a_2^2}$

$A = a_1 i + a_2 j + a_3 k$ $B = b_1 i + b_2 j + b_3 k$ $A+B = (a_1+b_1)i + (a_2+b_2)j + (a_3+b_3)k$

$|A| = \sqrt{a_1^2+a_2^2+a_3^2}$ } magnitude/norm $\vec{a} = |\vec{a}|\cos\alpha \Rightarrow \alpha = \cos^{-1}\left(\frac{\vec{a}}{|\vec{a}|}\right)$

Unit vector - a vector with magnitude 1 $\frac{\vec{a}}{|\vec{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$ since $\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{\vec{a}}{|\vec{a}|}$

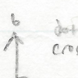
dot product - $A = \langle a_1, a_2, a_3 \rangle$ $B = \langle b_1, b_2, b_3 \rangle$ $A \cdot B = (a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3)$

↳ AKA scalar/inner product.

↳ $\underline{a}\cdot\underline{a} = |\underline{a}|^2$; $\underline{a}\underline{b} = \underline{b}\underline{a}$; $\underline{a}(\underline{b}+\underline{c}) = \underline{a}\underline{b} + \underline{a}\underline{c}$; $(K\underline{a})\underline{b} = K(\underline{a}\cdot\underline{b}) = \underline{a}(K\underline{b})$; $0 \cdot \underline{a} = 0$

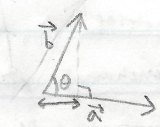
$\underline{a}\cdot\underline{b} = |\underline{a}|\cdot|\underline{b}|\cos\theta$ } the angle btw the vectors

$\underline{a}\cdot\underline{b} = 0 \Rightarrow$ 2 vectors are orthogonal



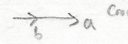
dot product = 0
cross product = $|\vec{a}||\vec{b}|$

Scalar projection $\text{comp}_a \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|a|} = |\vec{b}|\cos\theta$ (of $\vec{b}$ onto $\vec{a}$)



vector projection $\text{proj}_a \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|a|^2}\vec{a}$

dot product = $|\vec{a}||\vec{b}|$
cross product = 0



If $\vec{a},\vec{b}$ point in opposite directions, then $\theta = \pi$ so $\cos\theta = -1$ and $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$

If $\theta = 0$ or $\theta = \pi$ vectors are parallel.

Cross product - $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$, $\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

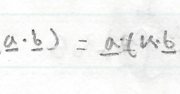
↳ AKA vector product, only ∃ for 3D vectors.

↳ $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ $\vec{a}\times\vec{b} = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}\vec{i} - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix}\vec{j} + \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\vec{k}$

↳ vector $a \times b$ orthogonal to $\vec{a}$ and $\vec{b}$

↳ if $0 \le \theta \le \pi$ for θ between $\vec{a}$ and $\vec{b}$ $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

↳ iff $\vec{a}\times\vec{b} = 0$ vectors are parallel. , $|\vec{a}\times\vec{b}| =$ area of parallelogram



↳ $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$; $(c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times(c\vec{b})$; $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$; $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c} + \vec{b}\times\vec{c}$;

$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$; $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})c$

$i \times j = k$ $j \times k = i$ $k \times i = j$

$j \times i = -k$ $k \times j = -i$ $i \times k = -j$

sphere w/ radius R centred at (a,b,c) $\qquad r\theta =$ arclength

$$(x-a)^2+(y-b)^2+(z-c)^2=R^2$$

parametric curves

$x=x(t)$
$y=y(t)$
$$\left.\frac{dy}{dx}\right|_{t=t_0}=\left.\frac{dy/dt}{dx/dt}\right|_{t=t_0} \qquad \frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \frac{d^2y}{dx^2}>0 \Rightarrow \text{concave up}, <0 \Rightarrow \text{concave down.}$$

If $\frac{dy}{dt}=0$, $\frac{dx}{dt}\neq 0$ horizontal tangent, $\frac{dy}{dt}\neq 0$, $\frac{dx}{dt}=0$ vertical tangent, if both, then $\lim_{t\to t_0}\frac{y(t)-y(t_0)}{x(t)-x(t_0)}$

$$A=\int_\alpha^\beta y(t)\frac{dx}{dt}(t)\,dt$$ curve must be traced once for $\alpha<t<\beta$, make sure x values are increasing or reverse signs.

$$L=\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$, if $L<0$ switch bounds.

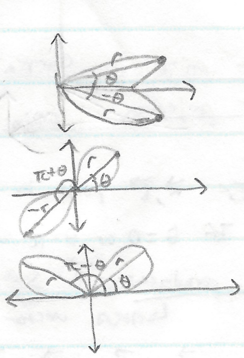
Cartesian $\leftrightarrow$ polar conversion

polar coordinates $P(r,\theta)$ $\quad x=r\cos\theta \quad r^2=x^2+y^2$
$\qquad y=r\sin\theta \quad \tan\theta=y/x$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{r'(\theta)\sin\theta+r(\theta)\cos\theta}{r'(\theta)\cos\theta-r(\theta)\sin\theta}\quad\left(\frac{dx}{d\theta}\neq 0\right)$$

$$A=\int\frac{1}{2}r^2\,d\theta \qquad L=\int_a^b\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\,d\theta$$

Symmetries



If unchanged after...	then symmetry about...
θ replaced with $-\theta$	polar axis
r replaced with $-r$ or θ replaced w/ $(\theta+\pi)$	the origin (pole)
θ replaced with $(\pi-\theta)$	the line $\theta/2$

$\hookrightarrow$ usually works when $r^2=f(\theta)$

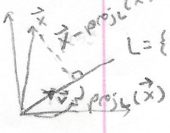
derivative of function w.r.t t.

$$(y-y_1)=m(x-x_1)$$

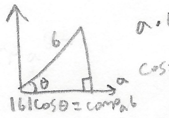
y_1: y of point, write as $y(t)$
x_1: x of point, write as $x(t)$

Find slope at solved t, becomes m, solve for x,y
$y=mx+b$

$\text{proj}_L(\vec{x}) = c\cdot\vec{v} = \frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$



$L=\{c\vec{v} : c\in\mathbb{R}\}$ we know $\text{proj}_L(\vec{x}) = c\vec{v} \Rightarrow (\vec{x}-\text{proj}_L(\vec{x}))\cdot\vec{v}=0 \Rightarrow (\vec{x}-c\vec{v})\cdot\vec{v}=0 \Rightarrow \vec{x}\cdot\vec{v}-c\vec{v}\cdot\vec{v}=0 \Rightarrow c=\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}$



$a\cdot b = |a||b|\cos\theta \Rightarrow |b|\cos\theta = \frac{a\cdot b}{|a|}$

$\cos\theta = \frac{adj}{hyp} = \frac{\text{comp}_a b}{|b|}$

$d = \frac{|\vec{QR}\times\vec{QP}|}{|\vec{QR}|}$ distance from point P to line w/ points Q, R on it...

distance btw planes $= \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$ $ax+by+cz=d_1$ and $ax+by+cz=-d_2$.

$\boxed{\text{vector equations}}$ - line has point (x_0, y_0, z_0) and direction $\underline{v}=\langle a,b,c\rangle$

↳ vector parametric equations – $\langle x_0, y_0, z_0\rangle + t\langle a,b,c\rangle$

↳ Scalar parametric equations – $x = x_0+ta,\ y=y_0+tb,\ z=z_0+tc.$

↳ Symmetric equations – (solve for t in each equation...) $t = \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

line segment from r_0 to r_1 = $\underline{r}(t) = (1-t)\underline{r}_0 + t\underline{r}_1$, for $0\le t\le 1$

can describe plane w/ a vectors and a point, cross product of these vectors gives n.

$\vec{PP_0}\cdot\underline{n}=0 \Rightarrow \langle x-x_0, y-y_0, z-z_0\rangle\cdot\langle a,b,c\rangle$

Skew lines - lines that do not intersect and are not parallel

↳ if parallel the direction vector $\langle a,b,c\rangle$ of one vector is a multiple of the other.

↳ if intersect can find t,s parameter value where parametric equations are equal.

Scalar equation of a plane – $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ plane through (x_0,y_0,z_0) w/ normal vector $\langle a,b,c\rangle$

linear equation of plane – $ax+by+cz+d=0$. plane w/ normal vector $\langle a,b,c\rangle$, $d=-(ax_0+by_0+cz_0)$ if a,b,c not all 0.

↳ 2 planes are parallel if their normal vectors are parallel.

→ angle between planes is angle btw the normal vectors to the plane $\cos\theta = \frac{n_1\cdot n_2}{|n_1||n_2|}$ $\theta\le 90°$.

→ line of intersection of both planes is $n_1\times n_2$, then set $z=0$ for both planes and solve $a_1x+b_1y+d_1=0$ and $a_2x+b_2y+d_2=0$ for x, y. this is finding point on line of intersection at $z=0$.

↳ or solve system of linear eqns in terms of x, y or z and use # as a parameter.

distance from point (x_1,y_1,z_1) to plane $ax+by+cz+d=0$: $= |ax_1+by_1+cz_1+d|/\sqrt{a^2+b^2+c^2} = |\text{comp}_n b| = \frac{|n\cdot b|}{|n|}$ $b=\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$ $n=\langle a,b,c\rangle$

$\boxed{\text{vector functions}}$ $\underline{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\underline{i}+g(t)\underline{j}+h(t)\underline{k} \Rightarrow x=f(t), y=g(t), z=h(t)$ } parametric eqn's with parameter t.

↳ domain is intersection of domains of $f(t), g(t)$ and $h(t)$ [whichever t's work in all 3 equations]

$\lim_{t\to a}\underline{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$ $= \lim_{h\to 0}\frac{\underline{r}(t+h)-\underline{r}(t)}{h}$ = tangent vector to curve defined by $\underline{r}$

Curve is continuous if $\lim_{t\to t_0}\underline{r}(t) = \underline{r}(t_0)$

$\underline{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$

1) $\frac{d}{dt}[\underline{u}(t)+\underline{v}(t)] = \underline{u}'(t)+\underline{v}'(t)$

2) $\frac{d}{dt}[c\underline{u}(t)] = c\underline{u}'(t)$

3) $\frac{d}{dt}[f(t)\underline{u}(t)] = f'(t)\underline{u}(t)+f(t)\underline{u}'(t)$

4) $\frac{d}{dt}[\underline{u}(t)\cdot\underline{v}(t)] = \underline{u}'(t)\cdot\underline{v}(t)+\underline{u}(t)\cdot\underline{v}'(t)$

5) $\frac{d}{dt}[\underline{u}(t)\times\underline{v}(t)] = \underline{u}'(t)\times\underline{v}(t)+\underline{u}(t)\times\underline{v}'(t)$

6) $\frac{d}{dt}[\underline{u}(f(t))] = \underline{u}'(f(t))f'(t)$

* if a curve lies on a sphere w/ centre origin, tangent vector $\underline{r}'(t)$ is always perpendicular to position vector $\underline{r}(t)$.

* angle of intersection of curves $\underline{r}_1(t), \underline{r}_2(t)$ at point t_0 is $\cos(\alpha) = \frac{\underline{r}_1'(t_0)\cdot\underline{r}_2'(t_0)}{|\underline{r}_1'(t_0)|\cdot|\underline{r}_2'(t_0)|}$

↳ angle btw tangent lines at the point of intersection.

$\int_a^b \underline{r}(t)\,dt = \underline{R}(b)-\underline{R}(a)$ R = antiderivative of $r(t)$ $\int_a^b \underline{r}(t) = \langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt\rangle = [F(t)\underline{i}+G(t)\underline{j}+H(t)\underline{k}+C]_a^b$

$\boxed{\text{function of 2 variables}}$ domain = D; range = $\{f(x,y) \mid (x,y)\in D\}$, graph: set of points in $\mathbb{R}^3$ for $x, y, z=f(x,y)$. $[(x,y)\in D]$

- level curves $f(x,y)=k$, sol'n is curve in xy plane. (k constant)

$\frac{\partial f}{\partial x}\int_y^x f(t)dt = f(x)$ $\frac{\partial f}{\partial y} = -f(y)$

Techniques of finding limits

DNE

- If 2 different limits come from 2 diff paths, limit DNE. — $\lim_{x\to a} f(x,b) \overset{?}{=} \lim_{y\to b} f(a,y)$; $\lim_{x\to a} f(x,x)$; $\lim_{x\to 0} f(x,x^2)$ (approaching at $y=x$, $y=x^2$)

- substitute let $t=g(x,y)$ and evaluate $\lim_{t\to *} f(t)$. (Can try polar coordinates...) — $\lim_{\substack{x\to a\\ y\to b}} f(g(x,y)) = \lim_{t\to *} f(t)$, $* = \lim_{\substack{x\to a\\ y\to b}} g(x,y)$. Can use basic limit laws... L'Hospital's rule...

- finding limit using basic limit laws... so long as the individual limits exist and $\neq 0$ in case of $\div$.
  - $\lim_{\substack{x\to a\\ y\to b}} \left(f \pm \times \div g\right)(x,y) = \lim_{\substack{x\to a\\ y\to b}} f(x,y) \pm \times \div \lim_{\substack{x\to a\\ y\to b}} g(x,y)$ [so long as $\lim g(x,y) \neq 0$ for $\div$]
  - $\lim_{\substack{x\to a\\ y\to b}} cf(x,y) = c \lim_{\substack{x\to a\\ y\to b}} f(x,y)$; $\lim_{\substack{x\to a\\ y\to b}} x = a$; $\lim_{\substack{x\to a\\ y\to b}} y = b$; $\lim_{\substack{x\to a\\ y\to b}} c = c$

- using squeeze theorem if $g(x,y) \le f(x,y) \le h(x,y)$ for all (x,y) near (a,b), and $\lim_{\substack{x\to a\\ y\to b}} g(x,y) = \lim_{\substack{x\to a\\ y\to b}} h(x,y) = L$, then $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L$ (try to isolate something like $0 \le \frac{x^2}{x^2+y^2} \le 1$) bounded...

- using continuity - if $f(x,y)$ is continuous at (a,b) $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$
  - polynomials and rational functions are continuous on their domains.

Techniques finding partial derivatives

- using definition - $\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$ - $\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$

- or use ordinary differentiation techniques... $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ (differentiated first...); $f_{yx} = \frac{\partial f}{\partial x \partial y}$ $f_{xy} = f_{yx} \Rightarrow$ Clairaut's theorem (must be defined and continuous at (a,b), then equal at a,b.

- implicit differentiation - let $z - f(x,y)$, treat x or y as variable, other fixed.

- f is differentiable at (x_0,y_0) if it has tangent plane at the point $(x_0, y_0, f(x_0,y_0))$, if $\exists\, a,b$ such that

$$\lim_{\substack{x\to x_0\\ y\to y_0}} \left(f(x,y) - \left[f(x_0,y_0) + a(x-x_0) + b(y-y_0)\right]\right) \Big/ \sqrt{(x-x_0)^2 + (y-y_0)^2} = 0$$

$z = f(x_0,y_0) + f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0)$ is the tangent plane.

f has tangent plane / linear approximation at (x_0,y_0) $\Leftrightarrow$ f differentiable at (x_0,y_0) $\Rightarrow$ $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist → partial derivatives can exist at point but not allways equal some value, so function not continuous at point (x_0,y_0)

f differentiable $\Rightarrow$ f continuous at (x_0,y_0)

f_x, f_y exist around (x_0,y_0) and continuous at (x_0,y_0) $\Rightarrow$ f differentiable

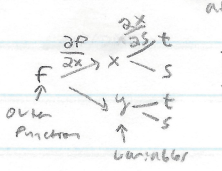
ex. $f(x,y) = \frac{xy}{x^2+y^2}$ → partial derivatives at point $(0,0)$ exists but function not continuous at $(0,0)$...

$\lim_{\substack{x\to 0\\ y\to 0}} f(x,0) = 0$

$\lim_{x\to 0} f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2} \neq 0$

Chain rule

use tree diagram.



$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$

tangent line formula: $y - y_0 = f'(x_0)(x-x_0)$